FLAT PENCILS OF SYMPLECTIC CONNECTIONS AND HAMILTONIAN OPERATORS OF DEGREE 2

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Abstract. Bi-Hamiltonian structures involving Hamiltonian operators of degree 2 are studied. Firstly, pairs of degree 2 operators are considered in terms of an algebra structure on the space of 1-forms, related to so-called Fermionic Novikov algebras. Then, degree 2 operators are considered as deformations of hydrodynamic type Poisson brackets.

1. Introduction

Hamilton's equations for a finite-dimensional system with position coordinates q^i and associated momenta p_i,

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial H}{\partial p_i}\,, \\ \frac{dp_i}{dt} &= -\frac{\partial H}{\partial q^i}\,, \end{aligned}$$

are understood geometrically as describing the flow of a vector field X_H which is associated with the Hamiltonian function $H(q^1,\ldots,q^n,p_1\ldots,p_n)$ by the formula $X_H(f)=\{f,H\}$, where $\{\cdot,\cdot\}$ is the Poisson bracket:

$$\{f,g\}=\sum_{i=1}^{n}\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right)\,. \tag{1}$$

More generally, one defines a Poisson bracket on an n-dimensional manifold M as a map $C^\infty(M)\times C^\infty(M)\to C^\infty(M)$, $(f,g)\mapsto\{f,g\}$, satisfying, for any functions f,g,h on M:

(1) antisymmetry: $\{f,g\}=-\{g,f\}$,
(2) linearity: $\{af+bg,h\}=a\{f,h\}+b\{g,h\}$ for any constants a,b,
(3) product rule: $\{fg,h\}=f\{g,h\}+g\{f,h\}$,
(4) Jacobi identity: $\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0$.

The conditions 1-3 identify $\{\cdot,\cdot\}$ as a bivector: a rank two, antisymmetric, contravariant tensor field ω on M. It can therefore be represented, by introducing coordinates $\{u^i\}$ on M, as a matrix of coefficients ω^{ij}, giving

$$\omega=\omega^{ij}\frac{\partial}{\partial u^i}\otimes\frac{\partial}{\partial u^j}=\frac{1}{2}\omega^{ij}\frac{\partial}{\partial u^i}\wedge\frac{\partial}{\partial u^j}\,,$$

and

$$\{f,g\}=\omega^{ij}\frac{\partial f}{\partial u^i}\frac{\partial g}{\partial u^j}\,. \tag{2}$$

The Jacobi identity places the following constraint on the components of ω:

$$\omega^{ir}\frac{\partial\omega^{jk}}{\partial u^r}+\omega^{jr}\frac{\partial\omega^{ki}}{\partial u^r}+\omega^{kr}\frac{\partial\omega^{ij}}{\partial u^r}=0\,. \tag{3}$$

If the matrix ω^{ij} is non-degenerate, we may introduce its inverse ω_{ij}, satisfying $\omega_{ir}\omega^{rj}=\delta_i^j$. The Jacobi identity for ω^{ij} is equivalent to the closedness of ω_{ij}.

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We refer to a closed non-degenerate two-form as a symplectic form, and a manifold equipped with one as a symplectic manifold. Darboux's theorem asserts that on any $2n$-dimensional symplectic manifold there exists a set of local coordinates $\{q^1,\ldots,q^n,p_1\ldots,p_n\}$ in which the Poisson bracket takes the form (1); i.e. the components of ω^{ij}, and so those of ω_{ij}, are constant.

One may also introduce Poisson brackets on infinite-dimensional manifolds. The loop space of a finite-dimensional manifold M, $L(M)$, is the space of smooth maps $u:S^1\to M$. Poisson brackets relating Hamiltonians to flows in $L(M)$ will therefore act on functionals mapping $L(M)\to\mathbb{R}$. In [5],[6] Dubrovin and Novikov studied the so-called Poisson brackets of differential-geometric type, which are of the form

$$\{f,g\}=\int \frac{\delta f}{\delta u^i}P^{ij}\left(\frac{\delta g}{\delta u^j}\right)dx \tag{4}$$

where u^i are coordinates on the target space M, and x is the coordinate on S^1. P^{ij} is a matrix of differential operators (in $\frac{d}{dx}$), with no explicit dependence on x, which is assumed to be polynomial in the derivatives $u^i_x, u^i_{xx},\ldots$. If $\{\cdot,\cdot\}$ defines a Poisson bracket on the loop space then P is referred to as a Hamiltonian operator.

There is a grading on such operators, preserved by diffeomorphisms of M, given by assigning degree 1 to $\frac{d}{dx}$, and degree n to the n^{th} x-derivative of each field u^i. An important class is the hydrodynamic type Poisson brackets, which are homogeneous of degree 1:

$$P^{ij}=g^{ij}(u)\frac{d}{dx}+\Gamma^{ij}_k(u)u^k_x\,.$$

According to the programme set out by Novikov [15], differential-geometric type Poisson brackets on $L(M)$ should be studied in terms of finite-dimensional differential geometry on the target space M. When expanded as a polynomial in $\frac{d}{dx}$ and the field derivatives, the coefficients, which are functions of the fields u^i alone, can often be naturally related to known objects of differential geometry, or else used to define new ones. In the hydrodynamic case, for instance, with g^{ij} non-degenerate, P is Hamiltonian if and only if g^{ij} is a flat metric on M and $\Gamma^k_{ij}=-g_{ir}\Gamma^{rk}_j$ are the Christoffel symbols of its Levi-Civita connection.

In [7] Dubrovin considered the geometry of bi-Hamiltonian structures of Hydrodynamic operators, that is pairs of such operators compatible in the sense of [13], that every linear combination of them also determines a Poisson bracket. In particular, he introduced a multiplication of covectors on M and expressed the compatibility of the operators in terms of a quadratic relations on this algebra.

This paper is principally concerned with Hamiltonian operators which are homogeneous of degree 2. Section 2 presents the differential geometry of such operators, and in particular relates the subclass which can be put into a constant form by a change of coordinates on M to symplectic connections. Section 3 then considers pairs of operators from this subclass, and the algebraic constraints their compatibility places upon the associated multiplication. In section 4 inhomogeneous bi-Hamiltonian structures consisting of a degree 1 and a degree 2 operator are studied.

2. Hamiltonian Operators of Degree 2

We begin with a review of known results on Hamiltonian operators of degree 2:

$$P^{ij}=a^{ij}\left(\frac{d}{dx}\right)^2+b^{ij}_k u^k_x\frac{d}{dx}+c^{ij}_{kl}u^k_xu^l_x+c^{ij}_k u^k_{xx}, \tag{5}$$

in which the matrix a^{ij} is assumed to be non-degenerate. Such operators have been considered already in, for example, [17], [14], [4], [15], in which the (conditional)

Darboux theorem has been discussed. In preparation for the bi-Hamiltonian theory we present these results without the use of special coordinates.

Under the change of coordinates $\tilde{u}^i = \tilde{u}^i(u^p)$ the coefficients in P^{ij} transform as

$$
\begin{aligned}
\tilde{a}^{ij} &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q} a^{pq}\,,\\
\tilde{b}^{ij}_k &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k} b^{pq}_r - 2\frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^s}{\partial u^q}\frac{\partial \tilde{u}^j}{\partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} a^{pq}\,,\\
\tilde{c}^{ij}_k &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k} c^{pq}_r - \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^s}{\partial u^q}\frac{\partial \tilde{u}^j}{\partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} a^{pq}\,,\\
\tilde{c}^{ij}_{kl} &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} c^{pq}_{rs} + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^j}{\partial u^q}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} c^{pq}_r\\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^{(k}}\frac{\partial u^s}{\partial \tilde{u}^{l)}} b^{pq}_r + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^3 \tilde{u}^j}{\partial u^q \partial u^r \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} a^{pq}\\
&\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} a^{pq}\,,
\end{aligned} \tag{6}
$$

where the brackets denote symmetrisation. So in particular a^{ij} transforms as a rank 2 contravariant tensor on the target space and b^{ij}_k and c^{ij}_k are related to Christoffel symbols of connections by $b^{ij}_k = -2a^{ir}\bar{\Gamma}^j_{rk}$ and $c^{ij}_k = -a^{ir}\Gamma^j_{rk}$. Call these connections $\bar{\nabla}$ and ∇ respectively.

The transformation rules for c^{ij}_{kl} are not determined uniquely by those for P, since (5) sees only the part symmetric in k and l. To fix c^{ij}_{kl}, we always assume the antisymmetric part is zero. Denote by a_{ij} the inverse of a^{ij} defined by $a_{ir}a^{rj} = \delta^j_i$.

The condition that the operation defined in (4) is skew-symmetric and satisfies the Jacobi identity places constraints on the coefficients appearing in (5).

Theorem 2.1. *The operator P in equation (5) defines a Poisson bracket by equation (4) if and only if*

(A) $a^{ij} = -a^{ji}$,
(B) $\nabla_k a^{ij} = b^{ij}_k - 2c^{ij}_k$,
(C) $a^{ir}\left(b^{jk}_r - 2c^{jk}_r\right) = a^{kr}\left(b^{ij}_r - 2c^{ij}_r\right)$,
(D) ∇ *is flat (zero torsion, zero curvature)*,
(E) $c^{ij}_{kl} = c^{ij}_{(k,l)} - a_{pr}c^{ri}_{(k}c^{pj}_{l)}$.

Proof. [14] states that, by virtue of being Hamiltonian, the operator (5) can be put in the form

$$P^{ij} = a^{ij}\left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx}\,, \tag{7}$$

by a change of coordinates $u^i = u^i(\tilde{u})$, and that for an operator of this shorter form to be Hamiltonian is equivalent to the three conditions

(a) $a^{ij} = -a^{ji}$,
(b) $a^{ij}{}_{,k} = b^{ij}_k$,
(c) $a^{ir}b^{jk}_r = a^{jr}b^{ki}_r$.

We first assume that P is a Poisson bracket, so there exists the special coordinates in which P takes the form (7) and (a)-(c) hold. By reversing the change of variables as $\tilde{u}^i = \tilde{u}^i(u)$, conditions (A)-(C) of Theorem 2.1 are Mokhov's three conditions converted to tensorial identities. That ∇ is flat follows from its Christoffel symbols, $\Gamma^k_{ij} = -a_{ir}c^{rk}_j$, being zero in the u coordinates.

The formula in condition (E) is derived from the transformation rules above. In changing from flat coordinates u^i to coordinates $\tilde{u}^i$ they give:

$$\begin{aligned}\tilde{c}^{ij}_{kl} &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^s}\frac{\partial u^s}{\partial \tilde{u}^{(k}}\frac{\partial u^s}{\partial \tilde{u}^{l)}} b^{pq}_r + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^3 \tilde{u}^j}{\partial u^q \partial u^r \partial u^s}\frac{\partial u^s}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} a^{pq} \\ &\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} a^{pq}\,,\end{aligned}$$

and

$$\begin{aligned}\tilde{c}^{ij}_k &= -\frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial \tilde{u}^s}{\partial u^q}\frac{\partial \tilde{u}^j}{\partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^s} a^{pq}\,, \\ &= \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial u^r}{\partial \tilde{u}^k} a^{pq}\,,\end{aligned}$$

where the last line has used the identity

$$\frac{\partial^2 \tilde{u}^i}{\partial u^r \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^j}\frac{\partial u^s}{\partial \tilde{u}^k} + \frac{\partial \tilde{u}^i}{\partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^j \partial \tilde{u}^k} = 0\,,$$

which is a differential consequence of $\frac{\partial \tilde{u}^i}{\partial u^r}\frac{\partial u^r}{\partial \tilde{u}^j} = \delta^i_j$.

$$\begin{aligned}\tilde{c}^{ij}_{k,l} &= \frac{\partial \tilde{c}^{ij}_k}{\partial \tilde{u}_l} \\ &= \frac{\partial^2 \tilde{u}^i}{\partial u^p \partial u^s}\frac{\partial u^s}{\partial \tilde{u}^l}\frac{\partial^2 \tilde{u}^j}{\partial u^r \partial u^q}\frac{\partial u^r}{\partial \tilde{u}^k} a^{pq} \\ &\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^3 \tilde{u}^j}{\partial u^q \partial u^r \partial u^s}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} a^{pq} \\ &\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial^2 u^r}{\partial \tilde{u}^k \partial \tilde{u}^l} a^{pq} \\ &\quad + \frac{\partial \tilde{u}^i}{\partial u^p}\frac{\partial^2 \tilde{u}^j}{\partial u^q \partial u^r}\frac{\partial u^r}{\partial \tilde{u}^k}\frac{\partial u^s}{\partial \tilde{u}^l} b^{pq}_s\,,\end{aligned}$$

from which we see

$$\tilde{c}^{ij}_{kl} = \tilde{c}^{ij}_{(k,l)} - \frac{\partial^2 \tilde{u}^i}{\partial u^p \partial u^s}\frac{\partial^2 \tilde{u}^j}{\partial u^r \partial u^q}\frac{\partial u^s}{\partial \tilde{u}^{(l}}\frac{\partial u^r}{\partial \tilde{u}^{k)}} a^{pq}\,.$$

This last term can be seen to be

$$\tilde{a}_{pr}\tilde{c}^{ri}_{(k}\tilde{c}^{pj}_{l)}\,.$$

Conversely, if (A)-(E) hold, the flatness of ∇ asserts the existence of coordinates in which $c^{ij}_k = 0$, and condition (E) then asserts that $c^{ij}_{kl} = 0$ in these coordinates. □

If we take, as a simple case, an operator P as in (5) with $b^{ij}_k = 2c^{ij}_k$ constants, and assume c^{ij}_{kl} to be defined by (E), then P is Hamiltonian if and only if $a^{ij} = A^{ij}_k u^k + A^{ij}_0$ where A^{ij}_k, A^{ij}_0 are constants with $A^{ij}_k = c^{ij}_k - c^{ji}_k$, $A^{ir}_l c^{jk}_r = A^{jr}_l c^{ik}_r$, $A^{ir}_0 c^{jk}_r = A^{jr}_0 c^{ik}_r$ and $c^{ij}_r c^{rk} + c^{ik}_r c^{rj}_l = 0$.

If we take an algebra $\mathcal{A}$ with basis $\{e^1, \ldots, e^n\}$, $n = \mathrm{dimM}$, and use c^{ij}_k and A^{ij}_0 to define a multiplication, $\circ$, and skew-symmetric bilinear form, $\langle\cdot,\cdot\rangle$, by $e^i \circ e^j = c^{ij}_r e^r$ and $\langle e^i, e^j\rangle = A^{ij}_0$, then we may rewrite these conditions as

$$\begin{aligned} e^i \circ e^j - e^j \circ e^i &= A^{ij}_r e^r\,, \\ (I \circ J) \circ K &= -(I \circ K) \circ J\,, && (8) \\ \Lambda(I,J,K) &= \Lambda(J,I,K)\,, && (9) \\ \text{and} \qquad \langle I, J \circ K\rangle &= \langle J, I \circ K\rangle\,, \end{aligned}$$

for all $I, J, K \in \mathcal{A}$, where Λ is the associator of $\circ$: $\Lambda(I,J,K) = (I \circ J) \circ K - I \circ (J \circ K)$.

Algebras satisfying conditions (8) and (9) have appeared before in [18], in the context of linear hydrodynamic Hamiltonian operators taking values in a completely odd superspace, where the following definition was proposed:

Definition 2.2. *An algebra $(\mathcal{A}, \circ)$ satisfying conditions (8) and (9) is called a Fermionic Novikov algebra.*

In [1] Fermionic Novikov algebras in dimensions 2-5 were studied, and the listing therein provides a source of examples of Hamiltonian operators of degree two.

Example 2.3.

$$
\begin{aligned}
P \;=\; & \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & -b-(t-1)u^1 \\ 0 & a & 0 & c-u^2 \\ -a & b+(t-1)u^1 & -c+u^2 & 0 \end{pmatrix} \left(\frac{d}{dx}\right)^2 \\
& +2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^1_x \\ 0 & 0 & -u^1_x & 0 \\ 0 & \tau u^1_x & u^2_x & u^3_x \end{pmatrix} \left(\frac{d}{dx}\right) \\
& + \left(\frac{1}{a}\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (u^1_x)^2 \\ 0 & 0 & -(u^1_x)^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u^1_x \\ 0 & 0 & -u^1_{xx} & 0 \\ 0 & \tau u^1_{xx} & u^2_{xx} & u^3_{xx} \end{pmatrix}
\end{aligned}
$$

is Hamiltonian for all values of the constants a, b, c and τ with $a \neq 0$. This is the most general Hamiltonian operator associated in the manner discussed above to the algebra designated $(44)_\tau$ in [1].

Returning to the general Hamiltonian operator (5), it can be seen from conditions (B) and (E) in Theorem 2.1 that the coefficients b^{ij}_k and c^{ij}_{kl} in (5) are completely determined by a^{ij} and c^{ij}_k. Thus the Hamiltonian operator on $L(M)$ is represented uniquely on M by only these latter two objects.

Theorem 2.4. *There is a one-to-one correspondence between Hamiltonian operators of the form (5) on $L(M)$ and pairs (a, ∇) on M consisting of a non-degenerate bivector a^{ij} and a torsion-free connection ∇ satisfying two conditions: firstly, that the curvature of ∇ vanishes, and secondly,*

$$a^{ir}\nabla_r a^{jk} = a^{jr}\nabla_r a^{ki}\,. \tag{10}$$

The Christoffel symbols, Γ^k_{ij}, of ∇ are related to c^{ij}_k by $c^{ij}_k = -a^{ir}\Gamma^j_{rk}$. We then have

$$
\begin{aligned}
b^{ij}_k &= \nabla_k a^{ij} + 2c^{ij}_k\,, \\
c^{ij}_{kl} &= c^{ij}_{k,l} - a_{pr}c^{ri}_{(k}c^{pj}_{l)}\,.
\end{aligned}
$$

With this, we may verify the following facts [17],[14]:

Corollary 2.5. *For P in (5) a Hamiltonian operator we have*

1. *Γ is the symmetric part of $\bar\Gamma$,*
2. *Let $\bar T^k_{ij} = \bar\Gamma^k_{ij} - \bar\Gamma^k_{ji}$ be the torsion of $\bar\nabla$. Then $\bar T_{ijk} = a_{ir}\bar T^r_{jk}$ is skew symmetric and the forms $\bar T = \frac{1}{6}\bar T_{ijk}du^i \wedge du^j \wedge du^k$ and $a = \frac{1}{2}a_{ij}du^i \wedge du^j$ are related by $3\bar T = da$.*

Proof. We begin by noting that equation (10) is equivalent to the condition

$$\nabla_k a_{ij} = \nabla_i a_{jk} \tag{11}$$

on the two-form a_{ij}.

In terms of covariant Christoffel symbols, Theorem 2.4 gives

$$\bar{\Gamma}^k_{ij} = \frac{1}{2} a^{kr} \nabla_r a_{ij} + \Gamma^k_{ij}\,, \tag{12}$$

from which it is clear that $\bar{\Gamma}^k_{(ij)} = \Gamma^k_{ij}$.

We therefore also have

$$\frac{1}{2} \nabla_k a_{ij} = \bar{\Gamma}_{ijk} - \Gamma_{ijk}\,,$$

where $\bar{\Gamma}_{ijk} = a_{ir}\bar{\Gamma}^r_{jk}$ and $\Gamma_{ijk} = a_{ir}\Gamma^r_{jk}$. Because ∇ is torsion-free we have

$$\begin{aligned}
\bar{T}_{ijk} &= \bar{\Gamma}_{ijk} - \bar{\Gamma}_{ikj}\,, \\
&= \bar{\Gamma}_{ijk} - \Gamma_{ijk} - \bar{\Gamma}_{ikj} + \Gamma_{ikj}\,, \\
&= \frac{1}{2}\nabla_k a_{ij} - \frac{1}{2}\nabla_j a_{ik}\,, \\
&= \nabla_k a_{ij}\,, \\
&= \nabla_{[k} a_{ij]}\,, \\
&= \frac{1}{3}(da)_{ijk}\,.
\end{aligned}$$

□

Lemma 2.6. *For a Hamiltonian operator of the form (5), the following three statements, presented in both covariant and contravariant forms, are equivalent:*

1. *The 2-form a is closed (and so symplectic), or equivalently a^{ij} satisfies equation (3) (and so defines a Poisson bracket on M by equation (2));*
2. $\nabla_k a^{ij} = 0$*, i.e.* $\nabla_k a_{ij} = 0$*;*
3. $b^{ij}_k = 2c^{ij}_k$*, i.e.* $\Gamma^k_{ij} = \bar{\Gamma}^k_{ij}$*.*

Proof. We see, from the characterisation of Hamiltonian operators given in Theorem 2.4,

$$\begin{aligned}
a^{ij} \text{ is Poisson} &\iff a^{ir}a^{jk}_{,r} + a^{jr}a^{ki}_{,r} + a^{kr}a^{ij}_{,r} = 0 \\
&\iff a^{ir}\nabla_r a^{jk} + a^{jr}\nabla_r a^{ki} + a^{kr}\nabla_r a^{ij} = 0 \\
&\iff 3a^{kr}\nabla_r a^{ij} = 0 \\
&\iff \nabla_k a^{ij} = 0\,, \\
&\iff b^{ij}_k = 2c^{ij}_k\,.
\end{aligned}$$

□

Lemma 2.6 therefore tells us that in the special case where the leading coefficient in P is the inverse of a symplectic form, the pair (a, ∇) defining P can be thought of as containing the symplectic form a_{ij}, and a torsionless connection compatible with it (in the sense that $\nabla a = 0$); that is, a symplectic connection. More precisely (see e.g. [3]):

Definition 2.7. *A symplectic connection on a symplectic manifold (M, ω) is a smooth connection ∇ which is torsion-free and compatible with the symplectic form ω, i.e.*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

and

$$(\nabla\omega)(X, Y, Z) = X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_Y Z) = 0\,,$$

where X, Y and Z are vector fields on M.

In local coordinates $\{x^i\}$, introducing Christoffel symbols Γ^k_{ij} for ∇ and writing $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$, the conditions for ∇ to be a symplectic connection read $\Gamma^k_{ij} = \Gamma^k_{ji}$, as usual, and

$$\nabla_k \omega_{ij} = \frac{\partial \omega_{ij}}{\partial x^r} - \Gamma^r_{ki}\omega_{rj} - \Gamma^r_{kj}\omega_{ir} = 0\,. \tag{13}$$

This definition is analogous to that of the Levi-Civita connection of a pseudo-Riemannian metric, however there is an important difference in that the Levi-Civita connection is uniquely specified by its metric. From the compatibility condition (13) it can be seen that if Γ^k_{ij} are the Christoffel symbols of a symplectic connection for ω, then the connection with Christoffel symbols $\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} + \omega^{kr} S_{rij}$ is a symplectic connection if and only if the tensor S_{ijk} is completely symmetric. In [10] a symplectic manifold with a specified symplectic connection is called, in light of [9], a Fedosov manifold. Here we call the pair (ω, ∇) of a symplectic form and a symplectic connection a Fedosov structure on M, and call the structure flat if ∇ is flat.

In the discussion of Hamiltonian operators it is convenient to work with contravariant quantities. We call

$$\Gamma^{ij}_k = -\omega^{ir}\Gamma^j_{rk}$$

the contravariant Christoffel symbols of the symplectic connection.

Result 2.8. *The compatibility of ∇ and ω is equivalent to*

$$\frac{\partial \omega^{ij}}{\partial x^k} = \Gamma^{ij}_k - \Gamma^{ji}_k\,.$$

Result 2.9. ∇ *being torsion-free is equivalent to* $\omega^{ir}\Gamma^{jk}_r = \omega^{jr}\Gamma^{ik}_r$.

The curvature of ∇,

$$R^k_{slt} = \partial_s \Gamma^k_{lt} - \partial_l \Gamma^k_{st} + \Gamma^k_{sr}\Gamma^r_{lt} - \Gamma^k_{lr}\Gamma^r_{st}\,,$$

can be expressed in terms of contravariant quantities by raising indices as

$$R^{ijk}_l = \omega^{is}\omega^{jt} R^k_{slt}\,.$$

This gives

Result 2.10.

$$R^{ijk}_l = \omega^{ir}\left(\partial_l \Gamma^{jk}_r - \partial_r \Gamma^{jk}_l\right) + \Gamma^{ij}_r \Gamma^{rk}_l + \Gamma^{ik}_r \Gamma^{rj}_l\,.$$

Having introduced symplectic connections, we are now in a position to interpret the following Darboux theorem for Hamiltonian operators of degree 2:

Theorem 2.11. [17] *Given a Hamiltonian operator*

$$P^{ij} = a^{ij}\left(\frac{d}{dx}\right)^2 + b^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl} u^k_x u^l_x + c^{ij}_k u^k_{xx}$$

where a^{ij} is non-degenerate, then P can be put in the constant form $P^{ij} = \omega^{ij}\left(\frac{d}{dx}\right)^2$ (where ω is a constant matrix) by a change of target space coordinates $\{u^i\}$ if and only if a_{ij} is closed. The coordinates in which this happens are flat coordinates for the connection $\Gamma^k_{ij} = -g_{ir} c^{rk}_j$ which can be chosen, using a linear substitution, to be canonical coordinates for the symplectic form $a_{ij} = \omega_{ij}$.

In arbitrary coordinates operators satisfying the conditions of Theorem 2.11 have the form

$$P^{ij} = \omega^{ij}\left(\frac{d}{dx}\right)^2 + 2\Gamma^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl} u^k_x u^l_x + \Gamma^{ij}_k u^k_{xx} \tag{14}$$

where ω^{ij} is the inverse of a symplectic form, $c^{ij}_{kl} = \Gamma^{ij}_{(k,l)} - \omega_{pr}\Gamma^{ri}_{(k}\Gamma^{pj}_{l)}$, and Γ^{ij}_k are the contravariant Christoffel symbols of a flat symplectic connection compatible with ω. This class of operators on $L(M)$ is therefore in one-to-one correspondence with flat Fedosov structures on M.

3. Flat Pencils of Fedosov Structures

In this section we consider pairs of Hamiltonian operators of the form (14):

$$\begin{aligned} P^{ij}_1 &= \omega^{ij}_1 \left(\frac{d}{dx}\right)^2 + 2{\Gamma_1}^{ij}_k u^k_x \frac{d}{dx} + {c_1}^{ij}_{kl} u^k_x u^l_x + {\Gamma_1}^{ij}_k u^k_{xx}\,, \\ P^{ij}_2 &= \omega^{ij}_2 \left(\frac{d}{dx}\right)^2 + 2{\Gamma_2}^{ij}_k u^k_x \frac{d}{dx} + {c_2}^{ij}_{kl} u^k_x u^l_x + {\Gamma_2}^{ij}_k u^k_{xx}\,. \end{aligned}$$

The first fact to establish is that if P_1 and P_2 are compatible then all elements of the pencil, $P_\lambda = P_1 + \lambda P_2$, remain in the class (14).

Theorem 3.1. *If P_1 and P_2 are compatible then ω^{ij}_1 and ω^{ij}_2 form a finite-dimensional bi-Hamiltonian structure on the target space.*

Proof. P_λ could have the general form

$$P^{ij}_\lambda = a^{ij}_\lambda \left(\frac{d}{dx}\right)^2 + {b_\lambda}^{ij}_k u^k_x \frac{d}{dx} + {c_\lambda}^{ij}_{kl} u^k_x u^l_x + {c_\lambda}^{ij}_k u^k_{xx}\,,$$

but clearly ${b_\lambda}^{ij}_k = 2{\Gamma_1}^{ij}_k + 2\lambda{\Gamma_2}^{ij}_k$ and ${c_\lambda}^{ij}_k = {\Gamma_1}^{ij}_k + \lambda{\Gamma_2}^{ij}_k$, so ${b_\lambda}^{ij}_k = 2{c_\lambda}^{ij}_k$, and hence, by Lemma 2.6, a^{ij}_λ satisfies the Jacobi identity (3) for all λ. □

So we write

$$P^{ij}_\lambda = \omega^{ij}_\lambda \left(\frac{d}{dx}\right)^2 + 2{\Gamma_\lambda}^{ij}_k u^k_x \frac{d}{dx} + {c_\lambda}^{ij}_{kl} u^k_x u^l_x + {\Gamma_\lambda}^{ij}_k u^k_{xx}\,.$$

An immediate corollary of Theorem 3.1 is that the tensor $L^i_j = \omega^{ir}_1 \omega_{2rj}$ has vanishing Nijenhuis torsion.

3.1. Multiplication of covectors.

As in [7], we proceed to understand the compatibility conditions on P_1 and P_2 in terms of the algebraic properties of a tensorial multiplication of covectors on M.

Definition 3.2. *Using the tensors*

$$\begin{aligned} \Delta^{sjk} &= \omega^{jr}_2 {\Gamma_1}^{sk}_r - \omega^{sr}_1 {\Gamma_2}^{jk}_r\,, \\ \Delta^{jk}_i &= \omega_{2is}\Delta^{sjk}\,, \end{aligned}$$

we define a multiplication $\circ$ of covectors on M by

$$(\alpha \circ \beta)_i = \alpha_j \beta_k \Delta^{jk}_i\,.$$

Theorem 3.3. *The compatibility of P_1 and P_2 is equivalent to*

$$\begin{aligned} (I, J\circ K)_2 &= (J, I \circ K)_2\,, &(15)\\ \text{and} \quad (I \circ J)\circ K &= 0\,, &(16) \end{aligned}$$

for all covectors I, J, K on M. Here $(\cdot,\cdot)_2$ is the skew-symmetric bilinear form on T^*M induced by ω^{ij}_2, i.e. $(I,J)_2 = I_r J_s \omega^{rs}_2$. The compatibility also implies

$$\nabla^2_l \Delta^{ij}_k = \nabla^2_k \Delta^{ij}_l\,. \qquad (17)$$

Because of Theorem 3.1, we phrase the compatibility of P_1 and P_2 in terms of Fedosov structures on M, and break the above theorem into stages:

Definition 3.4. *Two flat Fedosov structures (ω_1, ∇^1) and (ω_2, ∇^2), where ∇^1 and ∇^2 have contravariant Christoffel symbols ${\Gamma_1}^{ij}_k$ and ${\Gamma_2}^{ij}_k$ respectively, are said to be*

(i) almost compatible *if and only if $(\omega_\lambda, \nabla^\lambda)$ is a Fedosov structure for all λ, where the connection ∇^λ is given by ${\Gamma_\lambda}^{ij}_k = {\Gamma_1}^{ij}_k + \lambda {\Gamma_2}^{ij}_k$.*

(ii) almost compatible and flat *if and only if they are almost compatible, and in addition the curvature of ∇^λ vanishes for all λ.*

(iii) compatible *if and only if they are almost compatible and flat, and ${c_\lambda}^{ij}_{kl} = {\Gamma_\lambda}^{ij}_{(k,l)} - \omega_{\lambda pr} {\Gamma_\lambda}^{ri}_{(k} {\Gamma_\lambda}^{pj}_{l)}$ satisfies ${c_\lambda}^{ij}_{kl} = {c_1}^{ij}_{kl} + \lambda {c_2}^{ij}_{kl}$ for all λ.*

The compatibility of two flat Fedosov structures on M is equivalent to the compatibility of the associated Poisson brackets on $L(M)$.

We now turn to the two Fedosov strucutres defined by P_1 and P_2, and to the pair $(\omega_\lambda, \nabla^\lambda)$ defined by P_λ. From the linearity of Result 2.8 in the contravariant symbols it can be seen that ω_λ is automatically ∇^λ-constant, so the almost compatibility of (ω_1, ∇^1) and (ω_2, ∇^2) is equivalent to ∇^λ being torsion free, i.e. to

$$\omega_\lambda^{ir} {\Gamma_\lambda}^{jk}_l = \omega_\lambda^{jr} {\Gamma_\lambda}^{ik}_l \,.$$

In flat coordinates for ∇^2, this condition reduces to

$$\omega_2^{ir} {\Gamma_1}^{jk}_r = \omega_2^{jr} {\Gamma_1}^{ik}_r \,. \tag{18}$$

Note that we already have

$$\omega_1^{ir} {\Gamma_1}^{jk}_r = \omega_1^{jr} {\Gamma_1}^{ik}_r \,. \tag{19}$$

Lemma 3.5. *If (ω_1, ∇^1) and (ω_2, ∇^2) are almost compatible, then the flatness of ∇^λ is equivalent to either, and hence both, of*

$$\begin{aligned} \partial_l {\Gamma_1}^{jk}_s - \partial_s {\Gamma_1}^{jk}_l &= 0 & (20)\\ \textit{and} \quad {\Gamma_1}^{ij}_r {\Gamma_1}^{rk}_l + {\Gamma_1}^{ik}_r {\Gamma_1}^{rj}_l &= 0 & (21) \end{aligned}$$

in the flat coordinates for ∇^2.

Proof. The contravariant curvature of Γ_λ is

$$\begin{aligned} {R_\lambda}^{ijk}_l &= \omega_\lambda^{ir}\left(\partial_l {\Gamma_\lambda}^{jk}_r - \partial_s {\Gamma_\lambda}^{jk}_l\right) + {\Gamma_\lambda}^{ij}_r {\Gamma_\lambda}^{rk}_l + {\Gamma_\lambda}^{ik}_r {\Gamma_\lambda}^{rj}_l \\ &= {R_1}^{ijk}_l \\ &\quad + \lambda \Big\{ \omega_2^{is}\left(\partial_l {\Gamma_1}^{jk}_s - \partial_s {\Gamma_1}^{jk}_l\right) + \omega_1^{is}\left(\partial_l {\Gamma_2}^{jk}_s - \partial_s {\Gamma_2}^{jk}_l\right) \\ &\qquad + {\Gamma_2}^{ij}_r {\Gamma_1}^{rk}_l + {\Gamma_1}^{ij}_r {\Gamma_2}^{rk}_l + {\Gamma_1}^{ik}_r {\Gamma_2}^{rj}_l + {\Gamma_2}^{ik}_r {\Gamma_1}^{rj}_l \Big\} \\ &\quad + \lambda^2 {R_2}^{ijk}_l \,, \end{aligned}$$

which in flat coordinates for ${\Gamma_2}^{ij}_k$ reads

$$\begin{aligned} {R_\lambda}^{ijk}_l &= \omega_1^{ir}\left(\partial_l {\Gamma_1}^{jk}_r - \partial_r {\Gamma_1}^{jk}_l\right) + {\Gamma_1}^{ij}_r {\Gamma_1}^{rk}_l + {\Gamma_1}^{ik}_r {\Gamma_1}^{rj}_l \\ &\quad + \lambda \omega_2^{is}\left(\partial_l {\Gamma_1}^{jk}_s - \partial_s {\Gamma_1}^{jk}_l\right) . \end{aligned}$$

The vanishing of the order λ term is equivalent to equation (20), and with this the vanishing of the λ-independent term is equivalent to (21). □

Lemma 3.6. *If (ω_1, ∇^1) and (ω_2, ∇^2) are almost compatible then the condition ${c_\lambda}^{ij}_{kl} = {\Gamma_\lambda}^{ij}_{(k,l)} - \omega_{\lambda pr} {\Gamma_\lambda}^{ri}_{(k} {\Gamma_\lambda}^{pj}_{l)}$ reads, in the flat coordinates for ∇^2,*

$$ {\Gamma_1}^{ij}_r {\Gamma_1}^{rk}_l - {\Gamma_1}^{ik}_r {\Gamma_1}^{rj}_l = 0 \,. \tag{22}$$

Proof. For an arbitrary Fedosov structure (ω, ∇) the object $c^{ij}_{kl} = \Gamma^{ij}_{(k,l)} - \omega_{pr}\Gamma^{ri}_{(k}\Gamma^{pj}_{l)}$ can be converted into a quadratic expression in contravariant quantities as

$$\omega^{sk}c^{ij}_{kl} = \omega^{sk}\Gamma^{ij}_{(k,l)} - \frac{1}{2}\Gamma^{si}_{p}\Gamma^{pj}_{l} + \frac{1}{2}\Gamma^{pi}_{l}\Gamma^{sj}_{p} \,. \tag{23}$$

This has similarities to the formula for covariant curvature obtained in Result 2.10; only certain signs have changed. Indeed, if we define a quantity c^{j}_{rkl} by

$$c^{j}_{rkl}dx^r = \frac{1}{2}\left(\nabla_{\partial_k}\nabla_{\partial_l} + \nabla_{\partial_l}\nabla_{\partial_k}\right)dx^j \,,$$

then $c^{ij}_{kl} = \omega^{ir}c^{j}_{rkl}$.

We have two ways of expanding $\omega^{sk}_{\lambda}c_{\lambda}{}^{ij}_{kl}$, corresponding to whether we choose first to substitute it into equation (23), or to expand the pencil quantities. We work in flat coordinates for ∇^2; in these, $c_2{}^{ij}_{kl}$ also vanishes. First expanding the pencil we have

$$\begin{aligned}\omega^{sk}_{\lambda}c_{\lambda}{}^{ij}_{kl} &= \left(\omega^{sk}_1 + \lambda\omega^{sk}_2\right)c_1{}^{ij}_{kl} \,,\\ &= \omega^{sk}_1 c_1{}^{ij}_{kl} + \lambda\omega^{sk}_2 c_1{}^{ij}_{kl} \,,\end{aligned}$$

whilst (23) gives

$$\begin{aligned}\omega^{sk}_{\lambda}c_{\lambda}{}^{ij}_{kl} &= \omega^{sk}_{\lambda}\Gamma_{\lambda}{}^{ij}_{(k,l)} - \frac{1}{2}\Gamma_{\lambda}{}^{si}_{p}\Gamma_{\lambda}{}^{pj}_{l} + \frac{1}{2}\Gamma_{\lambda}{}^{pi}_{l}\Gamma_{\lambda}{}^{sj}_{p} \,,\\ &= \left(\omega^{sk}_1 + \lambda\omega^{sk}_2\right)\Gamma_1{}^{ij}_{(k,l)} - \frac{1}{2}\Gamma_1{}^{si}_{p}\Gamma_1{}^{pj}_{l} + \frac{1}{2}\Gamma_1{}^{pi}_{l}\Gamma_1{}^{sj}_{p} \,.\end{aligned}$$

The order 1 terms merely express equation (23) for P_1. Equality of the order λ terms is equivalent to $\Gamma_1{}^{ij}_{(k,l)} = c_1{}^{ij}_{kl}$ and so to

$$\begin{aligned}\omega^{sk}_1\Gamma_1{}^{ij}_{(k,l)} &= \omega^{sk}_1 c_1{}^{ij}_{kl} \,,\\ &= \omega^{sk}_1\Gamma_1{}^{ij}_{(k,l)} - \frac{1}{2}\Gamma_1{}^{si}_{p}\Gamma_1{}^{pj}_{l} + \frac{1}{2}\Gamma_1{}^{pi}_{l}\Gamma_1{}^{sj}_{p} \,.\end{aligned}$$

□

Proof of Theorem 3.3. Using equation (18) in Definition 3.2 it can be seen that in the flat coordinates for ∇^2 we have $\Delta^{ij}_k = \Gamma_1{}^{ij}_k$. Thus we may regard equations (18),(20),(21) and (22) as identities on Δ^{ij}_k; the result is Theorem 3.3. □

The condition imposed by equation (21) for an almost compatible and flat pair of Fedosov structures on the mutliplication $\circ$ is $(I \circ J) \circ K = -(I \circ K) \circ J$, i.e. the first condition (8) satisfied by the multiplication of a Fermionic Novikov algebra. In general (9) is not satisfied even for compatible Fedosov structures, however we do have, for two flat Fedosov structures, (ω_1, ∇^1), (ω_2, ∇^2), which are almost compatible,

$$\begin{aligned}&\omega^{ir}_1\nabla^2_r\Delta^{jk}_l - \omega^{jr}_1\nabla^2_r\Delta^{ik}_l\\ &\quad= \Delta^{ij}_r\Delta^{rk}_l - \Delta^{ir}_l\Delta^{jk}_r - \Delta^{ji}_r\Delta^{rk}_l + \Delta^{jr}_k\Delta^{ik}_r \,.\end{aligned}$$

So, in particular, if Δ^{ij}_k is constant in the flat coordinates for ∇^2, almost compatible and flat Fedosov structures will define a Fermionic Novikov algebra structure on the covectors of M.

In [1] it emerged that examples of such algebras which do not also satisfy the 'Bosonic' relation $(I \circ J) \circ K = (I \circ K) \circ J$, and hence $(I \circ J) \circ K = 0$, are relatively rare. ∇^2-constant multiplications arising from pairs of Fedosov structures which are almost compatible and flat, but not compatible, such as that given in Example 3.10 below, are in this class.

3.2. **The pencil in flat coordinates.** We now turn our consideration to the form the pencil takes in the flat coordinates for ∇^2. From the elements of the proof of Theorem 3.3 we have

$$P_\lambda^{ij} = \left(\omega_1^{ij} + \lambda\omega_2^{ij}\right)\left(\frac{d}{dx}\right)^2 + 2{\Gamma_1}_k^{ij} u_x^k \frac{d}{dx} + {\Gamma_1}_{k,l}^{ij} u_x^k u_x^l + {\Gamma_1}_k^{ij} u_{xx}^k \,. \tag{24}$$

The Jacobi identity for P_λ (without assuming P_1 and P_2 are Hamiltonian themselves) is equivalent to the constraints

(i) ω_2^{ij} is constant and antisymmetric,
(ii) ω_1^{ij} is antisymmetric,
(iii) $\omega_1^{ir}{\Gamma_1}_r^{jk} = \omega_1^{jr}{\Gamma_1}_r^{ik}$,
(iv) ${\omega_1^{ij}}_{,k} = {\Gamma_1}_k^{ij} - {\Gamma_1}_k^{ji}$,
(v) $\omega_2^{ir}{\Gamma_1}_r^{jk} = \omega_2^{jr}{\Gamma_1}_r^{ik}$,
(vi) ${\Gamma_1}_{k,l}^{ij} = {\Gamma_1}_{l,k}^{ij}$
(vii) ${\Gamma_1}_r^{ij}{\Gamma_1}_l^{rk} = 0$.

Proposition 3.7. *In a fixed coordinate system $\{u^i\}$ (the flat coordinates for Γ_2), given a constant non-degenerate 2-form ω_2^{ij} and a vector field $B = B^r\partial_r$ satisfying*

$$\left(\omega_2^{is}B_{,s}^r - \omega_2^{rs}B_{,s}^i\right)\omega_2^{jp}B_{,pr}^k = \left(\omega_2^{js}B_{,s}^r - \omega_2^{rs}B_{,s}^j\right)\omega_2^{ip}B_{,pr}^k \tag{25}$$

and

$$B_{,ir}^j\omega_2^{rs}B_{,sl}^k = 0 \tag{26}$$

then the prescription

$$\begin{aligned} \omega_1^{ij} &= -(\mathcal{L}_B\omega_2)^{ij} = \omega_2^{ir}B_{,r}^j - \omega_2^{jr}B_{,r}^i\,, \\ {\Gamma_1}_k^{ij} &= \omega_2^{ir}B_{,rk}^j \end{aligned}$$

satisfies the constraints (i)-(vii). *Further, all solutions of* (i)-(vii) *have this form.*

Proof. Equations (25) and (26) are the quadratic constraints, $\omega_1^{ir}{\Gamma_1}_r^{jk} = \omega_1^{jr}{\Gamma_1}_r^{ik}$ and ${\Gamma_1}_r^{ij}{\Gamma_1}_l^{rk} = 0$ respectively. That ω_1 and Γ_1 satisfy the (linear) constraints (iv), (v) and (vi) is an immediate consequence of their definition.

Using the Poincare lemma together with the symmetries expressed in conditions (vi) and (v), we have the existence of a vector field satisfying ${\Gamma_1}_k^{ij} = \omega_2^{ir}A^j{}_{,rk}$. With this condition (iv) gives $\omega_1^{ij} = -(\mathcal{L}_A\omega_2)^{ij} + c^{ij}$, where c^{ij} is a constant antisymmetric matrix. We may now introduce a vector field B with $B^i = A^i + \frac{1}{2}x^s {w_2}_{sr} c^{ri}$ which satisfies $\omega_1^{ij} = -\mathcal{L}_B\omega_2^{ij}$ and ${\Gamma_1}_k^{ij} = \omega_2^{ir}B_{,rk}^j$. □

Since ω_2 is a symplectic form, its symmetries are precisely (locally) Hamiltonian vector fields. Therefore, if ω_2 and ω_1 are given, the requirement that $\omega_1^{ij} = -\mathcal{L}_B\omega_2^{ij}$ fixes the non-Hamiltonian part of B. Then the condition ${\Gamma_1}_k^{ij} = \omega_2^{ir}B_{,rk}^j$ fixes the Hamiltonian to within a quadratic function. From the point of view of the multiplication of covectors from Section 3.1, the Hamiltonian affects only the commutative part of $\circ$, thus the anti-commutative part is fixed by ω_1^{ij} and ω_2^{ij}.

With consideration of the transformation rules (6), one can phrase Proposition 3.7 as the existence of a vector field B such that

$$\begin{aligned} \omega_1^{ij} &= -\mathcal{L}_B\omega_2^{ij}\,, \\ {\Gamma_1}_k^{ij} &= -\mathcal{L}_B{\Gamma_2}_k^{ij}\,. \end{aligned} \tag{27}$$

We can also calculate from (6) the correct interpretation of the Lie derivative for an object of type c^{ij}_{kl}, namely:

$$\begin{aligned} \mathcal{L}_X c^{ij}_{kl} &= X^r c^{ij}_{kl,r} - X^i_{,r} c^{rj}_{kl} - X^j_{,r} c^{ir}_{kl} + X^r_{,k} c^{ij}_{rl} + X^r_{,l} c^{ij}_{kr} \\ &\quad + X^r_{,kl} c^{ij}_r - \frac{1}{2} X^j_{rl} b^{ir}_k - \frac{1}{2} X^j_{,rk} b^{ir}_l - X^j_{,rkl} a^{ir} . \end{aligned}$$

If we work in the flat coordinates for Γ_2, so that the components ${c_2}^{ij}_{kl} = 0$, we have for our pencil

$$\begin{aligned} -\mathcal{L}_B {c_2}^{ij}_{kl} &= +\omega_2^{ir} B^j_{,rkl} , \\ &= (\omega_2^{ir} B^j_{,rk})_{,l} , \\ &= {\Gamma_1}^{ij}_{k,l} . \end{aligned}$$

Now, in the flat coordinates for ∇^2 we have the relation ${c_1}^{ij}_{kl} = {\Gamma_1}^{ij}_{k,l}$. The linearity of the transformation rules shows that the Lie derivative of ${c_2}^{ij}_{kl}$ should be an object of the same type as ${c_1}^{ij}_{kl}$. Thus we have, in addition to (27),

$${c_1}^{ij}_{kl} = -\mathcal{L}_B {c_2}^{ij}_{kl} .$$

One may understand these three infinitesimal relations between the coefficients of P_1 and P_2 as averring the existence on $L(M)$ of an evolutionary vector field

$$\hat{B} = B^i(u(x)) \frac{\partial}{\partial u^i(x)} + \dots$$

such that

$$P_1^{ij} = -\mathcal{L}_{\hat{B}} P_2^{ij} .$$

We now turn our attention to some examples of pairs of Fedosov structures, using the framework of Proposition 3.7.

Example 3.8. Two-dimensional pencils. *Without loss of generality we take*

$$\omega_2 = \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} ,$$

where u^1 and u^2 are a flat coordinate system for ∇^2.

We take

$$B = f(u^1, u^2) \frac{\partial}{\partial u^1} + g(u^1, u^2) \frac{\partial}{\partial u^2}$$

and from it calculate ω_1 and Γ_1 according to (27). In particular

$$\omega_1 = (f_{,1} + g_{,2}) \omega_2 ,$$

from which it follows immediately that (ω_1, ∇^1) and (ω_2, ∇^2) are almost compatible.

They are almost compatible and flat if and only if $h = f + \lambda g$ satisfies the homogeneous Monge-Ampere Equation $h_{12}^2 - h_{11} h_{22} = 0$ for all λ.

They are compatible if and only if $a = f + \lambda g$ and $b = f + \mu g$ satisfy

$$a_{12} b_{12} - a_{11} b_{22} = 0$$

for all λ, μ.

For instance, one may recover the three two-dimensional Fermionic Novikov algebras of [1] *as constant multiplications via*

(T1) $f = u^1$, $g = 0$,
(T2) $f = u^1$, $g = (u^1)^2$,
(T3) $f = (u^1)^2$, $g = 0$.

Example 3.9. Commutative algebras. *In the case in which ω_1 is constant in the flat coordinates for ∇^2, we have, by condition (iv),*

$$\Gamma_1{}^{ij}_k = \Gamma_1{}^{ji}_k \,,$$

so that the multiplication $\circ$ is commutative.

In particular if

$$\omega_1 = \omega_2 = \omega = \sum_{i=1}^n \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \,,$$

then the non-Hamiltonian part of B is

$$\sum_{i=1}^n q^i \frac{\partial}{\partial q^i} \,.$$

To this we may add a Hamiltonian vector field, giving

$$B = \sum_{i=1}^n \left(\left[q^i + \frac{\partial H}{\partial p_i} \right] \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) \,.$$

Since $\omega_1 = \omega_2$, equation (25) is immediate. Equation (26) becomes

$$H_{,ijr}\, \omega^{rs} H_{,skl} = 0 \,,$$

where the indices i, j, k, l, r, s account for both q and p variables.

A solution to this is $H = f(x^1, x^2, \ldots, x^n)$, where each x^i is either p_i or q^i; only one from each pair of conjugate variables features in H.

It is not hard to see that Proposition 3.7 can be modified to describe almost compatible and flat pairs of Fedosov structures. Specifically, we replace equation (26) by the expression corresponding to $\Gamma_1{}^{ij}_r \Gamma_1{}^{rk}_l = \Gamma_1{}^{ik}_r \Gamma_1{}^{rj}_l$, namely:

$$B^j_{,ir} \omega_2^{rs} B^k_{,sl} = B^j_{,lr} \omega_2^{rs} B^k_{,si} \,. \tag{28}$$

Example 3.10. *The Fedosov structures specified by*

$$\begin{aligned} \omega_2 &= \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q_2} \wedge \frac{\partial}{\partial p_2} \,, \\ \Gamma_2{}^{ij}_k &= 0 \,, \\ B &= \frac{3}{2} q_1^2 \frac{\partial}{\partial q_1} + 2 q_1 q_2 \frac{\partial}{\partial q_2} + q_1 p_2 \frac{\partial}{\partial p_2} \,, \end{aligned}$$

and $\omega_1^{ij} = -\mathcal{L}_B \omega_2^{ij}$ and $\Gamma_1{}^{ij}_k = -\mathcal{L}_B \Gamma_2{}^{ij}_k$ are almost compatible and flat, but not compatible.

The non-zero components of ω_1 and $\circ$ are

$$\begin{aligned} \{q_1, p_1\}_1 = \{q_2, p_2\}_1 &= 3q_1 \,, \\ \{q_2, p_1\}_1 &= 2q_2 \,, \\ \{p_2, p_1\}_1 &= p_2 \,, \end{aligned}$$

and

$$\begin{aligned} dq_2 \circ dp_2 &= dq_1 \,, \\ dp_1 \circ dq_1 &= -3dq_1 \,, \\ dp_1 \circ dq_2 &= -2dq_2 \,, \\ dp_1 \circ dp_2 &= -dp_2 \,, \\ dp_2 \circ dq_2 &= -2dq_1 \,. \end{aligned}$$

Thus, the products

$$\begin{array}{rrcl} & (dp_1 \circ dq_2) \circ dp_2 & = & -2dq_1 \\ \text{and} & (dp_1 \circ dp_2) \circ dq_2 & = & 2dq_1 \end{array}$$

violate equation (16) but not (8). Note that $\circ$ also satisfies (9) and thus defines a Fermionic Novikov algebra which is not 'Bosonic'.

3.3. ωN **manifold with Potential.** The tangent bundle T^*Q of a manifold Q is naturally equipped with a symplectic form, and thus cotangent bundles form the basic set of examples of symplectic manifolds. One may hope to find examples of finite-dimensional bi-Hamiltonian structures on cotangent bundles by exploiting the existence of additional structures on the underlying manifolds. The main object used to do this is a $(1,1)$-tensor L^i_j on Q whose Nijenhuis torsion is zero. Such an object was utilised by Benenti [2] to demonstrate the separability of the geodesic equations on a class of Riemannian manifolds. This result was later interpreted in [12] in terms of a bi-Hamiltonian structure on T^*Q which was extended to a degenerate Poisson pencil on $T^*Q \times \mathbb{R}$.

To obtain Fedosov structures we require more than just a tensor L^i_j on Q with vanishing Nijenhuis torsion; we also need a means of specifying the connections. If Q is equipped with a torsion-free connection $\widetilde{\nabla}$, then the Nijenhuis torsion of a $(1,1)$-tensor L^i_j can be written as

$$N^i_{jk} = L^s_j \widetilde{\nabla}_s L^i_k - L^s_k \widetilde{\nabla}_s L^i_j - L^i_s \widetilde{\nabla}_j L^s_k + L^i_s \widetilde{\nabla}_k L^s_j \,.$$

If there exists a vector field, A, on Q such that $L^i_j = \widetilde{\nabla}_j A^i$ then

$$N^i_{jk} = (\widetilde{\nabla}_j A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_k A^i) - (\widetilde{\nabla}_k A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_j A^i) - (\widetilde{\nabla}_s A^i)(R^s_{jkr} A^r)\,,$$

where R^i_{jkl} is the curvature tensor of $\widetilde{\nabla}$.

So, if $\widetilde{\nabla}$ is flat then the vanishing of the Nijenhuis tensor of $L = \widetilde{\nabla} A$ is equivalent to the identity

$$(\widetilde{\nabla}_j A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_k A^i) = (\widetilde{\nabla}_k A^s)(\widetilde{\nabla}_s \widetilde{\nabla}_j A^i)\,. \tag{29}$$

Proposition 3.11. *Given a manifold Q endowed with a flat connection $\widetilde{\nabla}$ and a vector field A satisfying (29), the cotangent bundle T^*Q is endowed with a compatible pair of Fedosov structures, (ω_1, ∇^1) and (ω_2, ∇^2), as follows: ω_2 is the canonical Poisson bracket on T^*Q.*

The connection ∇^2 on T^*Q is the horizontal lift [19] *of the connection $\widetilde{\nabla}$ on Q; i.e. the Christoffel symbols ${\Gamma_2}^k_{ij}$ of ∇^2 are zero in the coordinates induced on T^*Q by the flat coordinates for $\widetilde{\nabla}$.*

(ω_1, ∇^1) is calculated from (ω_2, ∇^2) according to the prescription of Proposition 3.7, where the vector field B is the horizontal lift of A to T^*Q.

Proof. Let $\{q^1, \ldots, q^n\}$ be flat coordinates for $\widetilde{\nabla}$ on Q, and $\mathcal{C} = \{q^1, \ldots, q^n, p_1, \ldots, p_n\}$ be the induced coordinates on T^*Q. Then

$$\omega_2 = \sum_{r=1}^{n} \frac{\partial}{\partial q^r} \wedge \frac{\partial}{\partial p_r}$$

and

$$B = \sum_{r=1}^{n} A^i \frac{\partial}{\partial q^i}\,.$$

The space of sections of the cotangent bundle of T^*Q, Ω, naturally splits into $\mathcal{P} = \mathrm{span}\{dp_i\}$ and $\mathcal{Q} = \mathrm{span}\{dq^i\}$. For ${\Gamma_1}^{ij}_k = \omega_2^{ir} B^j_{,rk}$ to be non-zero requires k to represent a variable q^k, and i to represent a p_i variable. Thus $\Omega \circ \Omega \subseteq \mathcal{Q}$ and

$\mathcal{Q} \circ \Omega = \{0\}$, meaning that $(\Omega \circ \Omega) \circ \Omega = \{0\}$. So the relation (26), ${\Gamma_1}^{ij}_r {\Gamma_1}^{rk}_l = 0$, is satisfied.

ω_1^{ij} has only one kind of non-zero component, $\omega^{p_i q^j} = A^j_{,i}$, so the expression $\omega_1^{ir} {\Gamma_1}^{jk}_r$ has only one non-zero case:

$$\sum_{x^r \in \mathcal{C}} \omega_1^{p_i x^r} {\Gamma_1}^{p_j q^k}_{x^r} = \sum_{r=1}^{n} \omega_1^{p_i q^r} {\Gamma_1}^{p_j q^k}_{q^r} = A^r{}_{,i} \, A^k{}_{,rj} \ ,$$

which is seen to be symmetric in i and j by condition (29), which in the flat coordinates q^i reads

$$A^s{}_{,j} \, A^i{}_{,sk} = A^s{}_{,k} \, A^i{}_{,sj} \ .$$

□

Example 3.12. *If the eigenvalues of $L : TQ \to TQ$ are functionally independent in some neighbourhood then they may be used as coordinates, and L takes the form*

$$L = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \otimes du^i \, .$$

In this case we may set $A = \sum_{i=1}^{n} \frac{1}{2}(u^i)^2 \frac{\partial}{\partial u^i}$, and have $\widetilde{\nabla}$ defined by vanishing Christoffel symbols in these coordinates.

This gives, writing v_i as the conjugate coordinate to u^i on T^*Q,

$$\begin{aligned}
\omega_2 &= \sum_{i=1}^{n} \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial v_i} \, , \\
\omega_1 &= \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i} \wedge \frac{\partial}{\partial v_i} \, , \\
{\Gamma_2}^{ij}_k &= 0 \\
{\Gamma_1}^{v_i u^i}_{u^i} &= -1 \, ,
\end{aligned}$$

and all other Christoffel symbols zero.

4. Bi-Hamiltonian Structures in Degrees 1 and 2

We now consider a pair of operators, P_1 and P_2 in which P_1 is a Hamiltonian operator of hydrodynamic type and P_2 is of second order, i.e.:

$$\begin{aligned}
P_1^{ij} &= g^{ij}(u) \frac{d}{dx} + \Gamma^{ij}_k(u) u^k_x \, , \\
P_2^{ij} &= a^{ij} \left(\frac{d}{dx} \right)^2 + b^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl} u^k_x u^l_x + c^{ij}_k u^k_{xx} \, ,
\end{aligned}$$

where g^{ij} is the inverse of a flat metric g_{ij} on M and $\Gamma^{ij}_k = -g^{ir}\Gamma^j_{rk}$ where the Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection of g. We also assume that P_2^{ij} is antisymmetric, so that $a^{ij} = -a^{ji}$, $b^{ij}_k = a^{ij}_{,k} + c^{ij}_k + c^{ji}_k$ and $c^{(ij)}_{kl} = c^{(ij)}_{(k,l)}$.

The motivation [8] for studying such pairs of operators comes not from regarding them as separate Hamiltonian operators, but from thinking of P_2^{ij} as a first order (dispersive) deformation of P_1^{ij} into some non-homogeneous Hamiltonian operator $P^{ij} = P_1^{ij} + \varepsilon P_2^{ij} + O(\varepsilon^2)$. Thus, in such a pair, it is sensible to regard the geometry of P_1^{ij} as being more intrinsic than any associated to P_2^{ij}.

We choose to work in flat coordinates for g so that g^{ij} is constant and $\Gamma^{ij}_k = 0$. Direct calculation of the Jacobi identity for P^{ij} in these coordinates yields

Theorem 4.1. *P_2 is an infinitesimal deformation of P_1, i.e. $P^{ij} = P_1^{ij} + \varepsilon P_2^{ij} + O(\varepsilon^2)$ satisfies the Jacobi identity to order ε, if and only if*

(I) $g^{ir}c_r^{jk} + g^{jr}c_r^{ik} = 0\,,$

(II) $c_{kl}^{ij} = c_{(k,l)}^{ij}\,,$

(III) $g^{ir}c_{l,r}^{jk} = g^{jr}(c_{l,r}^{ik} - c_{r,l}^{ik})\,,$

(IV) $g^{ir}(a_{,r}^{jk} - c_r^{jk}) + g^{jr}(a_{,r}^{ki} - c_r^{ki}) + g^{kr}(a_{,r}^{ij} - c_r^{ij}) = 0$

in the flat coordinates for g^{ij}.

By introducing the tensor $T_k^{ij} = a^{ir}\Gamma_{rk}^j + c_k^{ij}$ is it easy to convert conditions (I), (III) and (IV) to arbitrary coordinates, whilst condition (II) becomes

$$2c_{kl}^{ij} = c_{k,l}^{ij} + c_{l,k}^{ij} - c_k^{ri}\Gamma_{rl}^j - c_l^{ri}\Gamma_{rk}^j + T_r^{ij}\Gamma_{kl}^r + T_k^{rj}\Gamma_{rl}^i + T_l^{rj}\Gamma_{rk}^i\,.$$

To consider a bi-Hamiltonian structure involving operators P_1^{ij} and P_2^{ij} one need only add conditions (C), (D) and (E) of Theorem 2.1 to Theorem 4.1, however, condition (II) above allows (E) to be replaced by $c_r^{ij}c_l^{rk} = c_r^{ik}c_l^{rj}$.

Example 4.2. *As discussed in section 2, P_2 with $b_k^{ij} = 2c_k^{ij}$ constant and a^{ij} non-degenerate is Hamiltonian if and only if $a^{ij} = A_k^{ij}u^k + A_0^{ij}$ with $A_k^{ij} = c_k^{ij} = c_k^{ji}$, A_0^{ij} is constant, c_k^{ij} are the structure constants of a Fermionic Novikov algebra $(\mathcal{A}, \circ)$, and A_0^{ij} defines a skew-symmetric bilinear form on $\mathcal{A}$ satisfying $\langle I, J\circ K\rangle = \langle J, I\circ K\rangle$.*

If we ask that P_2 satisfies the above constancy conditions in the flat coordinates for g^{ij}, then, defining an inner product on $\mathcal{A}$ by $(e^i, e^j) = g^{ij}$, we have that the compatibility of P_1 and P_2 is equivalent to the additional constraints:

$$\begin{aligned}(I\circ J)\circ K &= (I\circ K)\circ J\,,\\ (I, J\circ K) &= -(J, I\circ K)\end{aligned}$$

and

$$(I,[J,K]) + (J,[K,I]) + (K,[I,J]) = 0\,,$$

where $[I,J] = I\circ J - J\circ I$ is the commutator of $\circ$, which is a Lie bracket by equation (9).

For example, if we take the algebra $(\mathcal{A} = \operatorname{span}\{e^1, e^2, e^3, e^4\}, \circ)$ where the only non-zero products are $e^3\circ e^3 = e^1$ and $e^4\circ e^3 = e^2$ then we may take as our symplectic form and metric

$$[\omega^{ij}] = \begin{pmatrix} 0 & 0 & a & b\\ 0 & 0 & b & c\\ -a & -b & 0 & d-u^2\\ -b & -c & -d+u^2 & 0\end{pmatrix}$$

and

$$[g^{ij}] = \begin{pmatrix} 0 & 0 & 0 & e\\ 0 & 0 & -e & 0\\ 0 & -e & f & g\\ e & 0 & g & h\end{pmatrix}\,,$$

for any choice of the constants a, b, c, d, e, f, g, h such that $e \neq 0$ and $b^2 \neq ac$.

This algebra, essentially $(57)_{-1}$, is the only algebra in [1] of dimension 2 or 4 which admits non-degenerate forms $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ satisfying the above compatibility conditions with $\circ$, other than the trivial case in which all products are zero, i.e. in which the Hamiltonian operators share the same flat connection, and so are simultaneously constant.

Proposition 4.3. *If P_2 is an infinitesimal deformation of P_1 then there exists a tensor field A^i_j such that*

$$\begin{aligned} a^{ij} &= g^{ir}A^j_r - g^{jr}A^i_r\,, \\ b^{ij}_k &= 2g^{is}A^j_{s,k} - g^{jr}A^i_{k,r} - g^{is}A^j_{k,s}\,, \\ c^{ij}_{kl} &= g^{is}A^j_{s,kl} - g^{is}A^j_{(k,l)s}\,, \\ c^{ij}_k &= g^{is}A^j_{s,k} - g^{is}A^j_{k,s} \end{aligned} \tag{30}$$

in flat coordinates for g^{ij}. Further, any (1,1)-tensor field A^i_j produces an infinitesimal deformation of P_1 by the above formulae.

Proof. Using the non-degeneracy of g^{ij}, we introduce objects θ^k_{ij} and ϕ_{ij} by

$$\begin{aligned} c^{ij}_k &= g^{ir}\theta^j_{rk}\,, \\ a^{ij} &= g^{ir}g^{js}\phi_{rs}\,. \end{aligned}$$

Then condition (I) of Theorem 4.1 is equivalent to $\theta^k_{ij} = -\theta^k_{ji}$, and so we regard θ^k_{ij} as a family of 2-forms θ^k indexed by k.

Condition (III) is equivalent to $\theta^k_{jl,i} = \theta^k_{il,j} - \theta^k_{ij,l}$, so that $d\theta^k = 0$ for each k. This allows us to introduce a family of 1-forms ψ^k such that

$$\theta^k_{ij} = (d\psi^k)_{ij} = \psi^k_{i,j} - \psi^k_{j,i}\,.$$

Each ψ^k can be adjusted by the addition of the exterior derivative, df^k, of some function f^k without affecting the value of θ^k_{ij}.

Writing $\alpha_{ij} = \phi_{ij} - g_{jr}\psi^r_i + g_{jr}\psi^r_k$, we find that condition (IV) is equivalent to the closedness of the 2-form α_{ij}, upon substituting ϕ_{ij} and ψ^i_j for a^{ij} and c^{ij}_k. Thus we may introduce a 1-form h with components h_i such that $\alpha_{ij} = h_{i,j} - h_{j,i}$, and so

$$\phi_{ij} = g_{jr}\psi^r_i - g_{jr}\psi^r_j + h_{i,j} - h_{j,i}\,.$$

If we now let $A^i_j = \psi^i_j + (g^{ir}h_r)_{,j}$ then we have $\theta^k_{ij} = A^k_{i,j} - A^k_{j,i}$ and $\phi_{ij} = g_{jr}\psi^r_i - g_{ir}\psi^r_j$, so that the two equations $a^{ij} = g^{ir}A^j_r - g^{jr}A^i_r$ and $c^{ij}_k = g^{ir}A^j_{r,k} - g^{jr}A^j_{k,r}$ are satisfied. The remaining to equations follow easily from $c^{ij}_{kl} = c^{ij}_{k,l}$ and $b^{ij}_k = a^{ij}_k + c^{ij}_k + c^{ji}_k$.

For the converse, it is easy to check that conditions (I)-(IV) of Theorem 4.1 follow from (30) for any tensor field A^i_j. $\square$

As with Proposition 3.7, Proposition 4.3 may be understood as asserting the existence of an evolutionary vector field

$$e = A^i_j\,(u(x))\,u^j_x(x)\frac{\partial}{\partial u^i(x)} + \dots$$

satisfying $P_2 = -\mathcal{L}_e P_1$ whenever P_2 is an infinitesimal deformation of P_1. This is therefore not a surprising result; in [11] Getzler showed the triviality of infinitesimal deformations of Hydrodynamic type Poisson brackets. With this, Proposition 4.3 can be looked upon as a proof of Theorem 4.1.

There is a freedom in A^i_j of $A^i_j \mapsto A^i_j + g^{ir}f_{,rj}$ for some function f, which does not affect the coefficients of P_2. This corresponds to adjusting e by a Hamiltonian vector field, $e \mapsto e + P_1(\delta f)$.

If, with reference to Lemma 2.6, we impose the additional constraint on (30) that $b^{ij}_k = 2c^{ij}_k$ then we have the potentiality condition $g_{jr}A^r_{k,i} = g_{ir}A^r_{k,j}$, so that there exists a 1-form B_k such that

$$A^i_j = g^{ir}B_{j,r}\,. \tag{31}$$

In this case $a^{ij} = g^{ir}g^{jr}(B_{r,s} - B_{s,r}) = g^{ir}g^{jr}(dB)_{rs}$ and the freedom $A^i_j \mapsto A^i_j + g^{ir}f_{,rj}$ is $B \mapsto B + df$. This means that B can be determined purely from g^{ij} and a^{ij}, and thus there is no freedom in the choice of c^{ij}_k and c^{ij}_{kl}. In fact we may write explicitly

$$c^{ij}_k = g^{js}g_{kr}\frac{\partial a^{ir}}{\partial u^s}\,, \quad c^{ij}_{kl} = c^{ij}_{(k,l)}\,, \tag{32}$$

and with this, P_2 is an infinitesimal deformation of P_1 if and only if

$$g^{ir}a^{jk}_{,r} + g^{jr}a^{ki}_{,r} + g^{kr}a^{ij}_{,r} = 0\,. \tag{33}$$

Corollary 4.4. *Given a flat metric g and a symplectic form ω, there is at most one choice of flat symplectic connection ∇ such that the degree 2 Hamiltonian operator specified by (ω, ∇) is compatible with the hydrodynamic operator specified by g.*

Clearly, if this connection exists it is given by(32), so this definition must be checked against Theorem 2.1 to verify

$$P^{ij}_2 = \omega^{ij}\left(\frac{d}{dx}\right)^2 + 2c^{ij}_k u^k_x \frac{d}{dx} + c^{ij}_{kl}u^k_x u^l_x + c^{ij}_k u^k_{xx}$$

is Hamiltonian. Since equation (33) is a consequence of the antisymmetry of P_2, compatibility with the Hydrodynamic operator follows immediately.

We conclude this section with an example of this type.

Example 4.5. *The Kaup-Broer system* [16],

$$\begin{pmatrix} u^1_t \\ u^2_t \end{pmatrix} = \begin{pmatrix} u^1_{xx} + 2u^2_x + 2u^1u^1_x \\ -u^2_{xx} + 2(u^1u^2)_x \end{pmatrix},$$

is described by the pair of compatible Hamiltonian operators

$$\begin{aligned} P_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx}\,, \\ P_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\frac{d}{dx}\right)^2 + \begin{pmatrix} 2 & u^1 \\ u^1 & 2u^2 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} 0 & u^1_x \\ 0 & u^2_x \end{pmatrix}. \end{aligned}$$

Scaling $x \mapsto \varepsilon x$, $t \mapsto \varepsilon t$ splits P_2 into $P^{(1)}_2 + \varepsilon P^{(2)}_2$ where

$$\begin{aligned} P^{(1)}_2 &= \begin{pmatrix} 2 & u^1 \\ u^1 & 2u^2 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} 0 & u^1_x \\ 0 & u^2_x \end{pmatrix}, \\ P^{(2)}_2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\frac{d}{dx}\right)^2. \end{aligned}$$

Since $P_2 = P^{(1)}_2 + \varepsilon P^{(2)}_2$ is Hamiltonian for all ε, $P^{(1)}_2$ and $P^{(2)}_2$ constitute a bi-Hamiltonian structure of the type considered above. A set of flat coordinates for the metric in $P^{(1)}_2$ is

$$\begin{aligned} \tilde{u}^1 &= u^1\,, \\ \tilde{u}^2 &= \sqrt{4u^2 - (u^1)^2}\,, \end{aligned}$$

in which

$$\begin{aligned} \tilde{P}^{(1)}_2 &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\frac{d}{dx}\,, \\ \tilde{P}^{(2)}_2 &= \frac{2}{\tilde{u}^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\frac{d}{dx}\right)^2 + \frac{4}{(\tilde{u}^2)^2}\begin{pmatrix} 0 & -\tilde{u}^2_x \\ 0 & \tilde{u}^1_x \end{pmatrix}\frac{d}{dx} \\ &\quad + \frac{4}{(\tilde{u}^2)^3}\begin{pmatrix} 0 & (\tilde{u}^2_x)^2 \\ 0 & -\tilde{u}^1_x\tilde{u}^2_x \end{pmatrix} + \frac{2}{(\tilde{u}^2)^2}\begin{pmatrix} 0 & -\tilde{u}^2_{xx} \\ 0 & \tilde{u}^1_{xx} \end{pmatrix}. \end{aligned}$$

So in this situation we have, for the 1-form in (31),

$$B = \frac{\tilde{u}^1}{2\tilde{u}^2} d\tilde{u}^2 .$$

5. Conclusions

In section 3 an approach was taken based upon the methods of [7] to study compatible pairs of Hamiltonian operators of degree 2 which satisfy the conditions of the relevant Darboux theorem, Theorem 2.11. As for Hydrodynamic Poisson pencils, the compatibility could be reduced to algebraic constraints on a multiplication of covectors. Driving this was the ability to reduce a given Hamiltonian operator on $L(M)$ to a flat Fedosov structure (ω, ∇) on M, which are natural symplectic analogues of the pair consisting of a flat metric and its Levi-Civita connection which determines a Hydrodynamic Poisson bracket.

To extend such a results to pairs of arbitrary degree 2 Hamiltonian operators, one must consider the pair (a, ∇) of Theorem 2.4. The condition (10), whilst atypical, expresses a familiar concept; in almost-symplectic geometry, it is common to consider connections such that the covariant derivative of the almost-symplectic form is zero, but which have torsion; if the torsion of such a connection is skew-symmetric then its symmetric part satisfies (10). Equation (12) provides the means of going from the symmetric connection to the compatible connection with skew-torsion. The only formula missing above necessary to the study of arbitrary bi-Hamiltonian structures of degree 2 is an expression for the contravariant curvature of the connection defined by c^{ij}_k, which is, in the presence of Theorem 2.1's condition (B),

$$R^{ijk}_l = a^{ir}(c^{jk}_{r,l} - c^{jk}_{l,r}) + c^{ij}_r c^{rk}_l + c^{ik}_r c^{rj}_l - (b^{ij}_r - 2c^{ij}_r)c^{rk}_l + c^{ik}_r(b^{rj}_l - 2c^{rj}_l) .$$

One may use (B) to replace the components of b^{ij}_k in this expression with those of c^{ij}_k and the derivatives of a^{ij}. However, one sees that the compatibility conditions do not naturally become algebraic constraints on Δ^{ij}_k, and the relevancy of such an approach is undermined. It is interesting to note, however, that equation (23) still holds (with $\Gamma^{ij}_k = c^{ij}_k$), so that $\circ$ defined by Δ^{ij}_k still satisfies $(I \circ J) \circ K = (I \circ K) \circ J$, and that it is the 'Fermionic' condition $(I \circ J) \circ K = -(I \circ K) \circ J$ which is altered.

The proof of Proposition 3.7 is easily adapted to confirm the existence of a vector field B realising $P_1 = -\mathcal{L}_B P_2$ whenever P_1, of the form (5) is an infinitesimal deformation of P_2 as a Hamiltonian operator, provided ${b_1}^{ij}_k = 2{c_1}^{ij}_k$. A simple calculation of $\mathcal{L}_B P_2$ for arbitrary B shows that ${b_1}^{ij}_k = 2{c_1}^{ij}_k$ is also a necessary condition. Thus we have determined the trivial deformations of a degree 2 Hamiltonian operator admitting a constant form, which are themselves of degree 2. Clearly a different approach is necessary to understand deformations of higher degrees. For the case of operators not satisfying the constraints of Theorem 2.11, it is not immediately obvious what conditions, if any, will guarantee the triviality of a deformation; owing to the different form the contravariant curvature tensor takes, the condition ${c_1}^{ij}_{k,l} = {c_1}^{ij}_{l,k}$ is absent. Owing to the lack of a constant form, the methods of [8] in ascertaining the triviality of higher degree deformations, if applicable, will be somewhat more complicated.

Finally, there is a certain artificiality to the examples of compatible Fedosov structures presented in section 3. Given Theorem 3.1's assertion that underlying a pair of compatible Fedosov structures is a finite-dimensional bi-Hamiltonian structure, the question is raised asking which finite-dimensional bi-Hamiltonian structures admit symplectic connections forming almost compatible, almost compatible and flat, or compatible Fedosov structures? It would be interesting to exhibit a

pair of compatible Fedosov structures in which the flat coordinates for one of the connections are in some sense physical.

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